Large N Scaling Behavior of the Lipkin-Meshkov-Glick Model

F. Leyvraz[1] and W.D. Heiss[2]

[1] *Centro de Ciencias Físicas, Universidad Nacional Autónoma de Mexico, Cuernavaca, 62551 Morelos, Mexico*
[2] *Institute of Theoretical Physics and Department of Physics, University of Stellenbosch, 7602 Matieland, South Africa*

(Dated: February 1, 2008)

We introduce a novel semiclassical approach to the Lipkin model. In this way the well-known phase transition arising at the critical value of the coupling is intuitively understood. New results – showing for strong couplings the existence of a threshold energy which separates deformed from undeformed states as well as the divergence of the density of states at the threshold energy – are explained straightforwardly and in quantitative terms by the appearance of a double well structure in a classical system corresponding to the Lipkin model. Previously unnoticed features of the eigenstates near the threshold energy are also predicted and found to hold.

PACS numbers: 03.65.Sq, 05.30.Fk, 64.60.-i

The Lipkin-Meshkov-Glick (LMG) model, originally introduced in nuclear physics has found applications in a broad range of other topics: statistical mechanics of quantum spin systems [2], Bose–Einstein condensates [3] as well as quantum entanglement [4], to name but a few. The continued interest in this system arises from the fact that it is an exactly solvable [5, 6] many-body interacting quantum system as well as one of the simplest to show a quantum transition in the regime of strong coupling.

This transition is by now well-understood: the ground state becomes degenerate and a macroscopic change in the ground state energy takes place. Furthermore, at the transition value of the coupling, the density of states diverges at the ground state energy as the number N of interacting particles becomes large. The energy gap was recently found [7] to vanish as $N^{-1/3}$. Furthermore, in this model a novel type of phase transition has recently been discovered for strong values of the coupling parameter. Indeed, in this regime the spectrum is divided by a *critical energy* E_c, where the behavior characteristic of strong coupling holds below E_c, while above E_c the system reverts to the kind of behavior found below the phase transition. At the critical energy, the density of states is again found to diverge as the particle number N goes to infinity. This divergence has been conjectured to be of the logarithmic type.

In recent papers [7, 8] different approaches have been used to explain these results. The continuous unitary transformation technique (flow equations) [7, 9] was applied to obtain the spectrum reliably for large N values. An investigation of the singularities of the spectrum [8] in the complex λ-plane [10] (the exceptional points [11]) sheds more light upon the complexity of the limit problem but a final answer about the limit attained has not been given. These singularities have been recognized as an essential mechanism to invoke the phase transition in a seminal paper by Lee and Yang [12] and their significance for the partition function is discussed more recently again in [13, 14]. A different approach [7] starts with the bosonization method using higher orders of powers in $1/N$ of the Holstein-Primakoff representation and then applies the flow equation technique. New results have been obtained in this way, in particular the correct analytic behavior of the level distance as a function of N at the critical point. Earlier attempts [15] have revealed different results such as the form of the wave functions beyond the phase transition.

In this Letter we introduce a novel semiclassical approach to the LMG model. It readily explains the above features, determines the precise value of the critical energy as a function of the coupling, proves the logarithmic divergence of the density of states near the critical energy as well as successfully predicts certain previously unnoticed behavior of the eigenstates near the critical energy. Finally, we obtain a qualitative understanding of all essentials of the model: the classical model we introduce has a double well structure above the phase transition, and the critical energy can then be identified with the separatrix energy. The approach given here also shows easily both the nature of the phase transition as a function of the coupling parameter as well as the scaling with N of the vanishing gap at the critical coupling, which was previously shown [7] to scale as $N^{-1/3}$.

We here recapitulate the basics of the model and discuss the essential properties for large values of N. It is given in terms of $2j+1 = N+1$-dimensional representations of the SU(2) operators J_k, $k = x, y, z$ as follows

$$H(\lambda) = J_z + \frac{\lambda}{N}(J_x^2 - J_y^2). \tag{1}$$

Here the interaction is scaled by N to ensure that H is extensive. In this form the model has a phase transition just beyond $\lambda = 1$, the larger N the closer the transition point at $\lambda = 1$. This has been discussed under various points of view in the literature, see e.g. [5, 16].

The Hamiltonian allows reduction into two spaces: m integer and m half-integer, with m the eigenvalues of J_z;

it corresponds to N even and odd respectively and is denoted as parity. For $\lambda \gtrsim 0$ the even and odd levels are obviously separated and remain so for all $\lambda < 1$ while the levels become degenerate (up to terms vanishing exponentially fast in N) for $\lambda > 1$. The phase at $\lambda < 1$ is called the normal phase, while the symmetry (parity) breaking phase at $\lambda > 1$ is called the deformed phase. Recent calculations [9] apply non-perturbative flow equations allowing to obtain the spectrum for arbitrarily high yet finite values of N. These have established [8] the existence of the phase transition in energy referred to above: the states having energy below a certain threshold behave as states of the deformed phase, whereas higher in the spectrum the states become undeformed again.

Since the commutator $[H(\lambda), (\vec{J})^2]$ vanishes, we confine ourselves to a fixed value of $j = N/2$. For large N we consider the Hamiltonian (1) on the sphere of radius $j = N/2$. In other words, we rewrite the Hamilton operator (1) as a classical Hamilton function

$$H = \frac{N}{2}\left(-\sin\theta\cos\phi - \frac{\lambda}{2}(\cos^2\theta - \sin^2\theta\sin^2\phi)\right) \quad (2)$$

having introduced the polar angles as [17]

$$\begin{aligned} J_z &= -\frac{N}{2}\sin\theta\cos\phi \\ J_x &= \frac{N}{2}\cos\theta \\ J_y &= \frac{N}{2}\sin\theta\sin\phi. \end{aligned}$$

Note that the transition to a classical Hamiltonian has also been achieved in a different way using the coherent state approach developed in [18]. At this point we notice that, with $\mu = \cos\theta$, the Poisson bracket

$$\{\mu, \phi\} = \frac{2}{N} \quad (3)$$

suggests how to quantize the Hamilton function of the single particle problem in the two canonical conjugate coordinates ϕ and μ. It can be written as

$$K \equiv \frac{2H}{N} = -\sqrt{1-\mu^2}\cos\phi - \frac{\lambda}{2}(\mu^2 - (1-\mu^2)\sin^2\phi). \quad (4)$$

To obtain information about the ground state and the low lying states we expand $2H/N$ around its minimum which is found at

$$\sin\phi_0 = 0 \quad (5)$$

$$\mu_0 = \begin{cases} 0 & (\lambda \le 1) \\ \pm\sqrt{1-\lambda^{-2}} & (\lambda \ge 1) \end{cases} \quad (6)$$

and the corresponding minimum values of H at

$$H = \begin{cases} -\frac{N}{2} & (\lambda \le 1) \\ -\frac{N}{4}(\lambda + \lambda^{-1}) & (\lambda \ge 1) \end{cases} \quad (7)$$

Expanding around the minimum reveals in fact all essential features. Keeping to lowest order terms we obtain around $\mu = \phi = 0$

$$K = -1 + \frac{1+\lambda}{2}\phi^2 + \frac{1-\lambda}{2}\mu^2 + \frac{\mu^4}{8} + \dots \quad (8)$$

In this form, quantization is straightforward. Based on (3) we identify ϕ and μ with momentum and position, respectively, and $2/N$ with $\hbar$, i.e. we use the usual canonical commutation relations for μ and ϕ. With this identification the Hamiltonian (8) represents a quartic oscillator which behaves for $\lambda < 1$ basically like a traditional oscillator with a harmonic spectrum (for the lower states) $E_k \sim k\sqrt{1-\lambda^2}$, $k = 1, 2, \dots$. For $\lambda > 1$, the lower states must be determined at the minimum in μ around the values μ_0 as given in (6). With $\epsilon = \mu - \mu_0$ the expansion of the Hamiltonian K yields, up to the constant given in (7)

$$K = \frac{\lambda^2 - 1}{2}\lambda\epsilon^2 + \frac{\phi^2}{\lambda}. \quad (9)$$

For low-lying levels, the harmonic spectrum is again obtained from (9) with frequency $\sqrt{2(\lambda^2-1)}$ well-known from previous work [5, 19].

To evaluate the average energy density, the features of which have been the object of recent work [8], we exploit, as we are in the semi-classical regime, the WKB relation

$$S(2E_k/N) = 2\pi\left(k + \frac{1}{2}\right)\hbar. \quad (10)$$

Here $S(E)$ denotes the action corresponding to the Hamiltonian K. By differentiation we obtain

$$\Delta E = E_{k+1} - E_k = \frac{\pi\hbar N}{T(2\overline{E}/N)} = \frac{2\pi}{T(\overline{E})} \quad (11)$$

where $T(E)$ is the period of the orbit with respect to K as a function of energy and $\overline{E}$ is $(E_{k+1} + E_k)/2$. Use is made of $T(E) = dS/dE$.

The last relation (11) contains virtually all basic information of the Lipkin model. Firstly, for $\lambda < 1$ nothing dramatic happens: the density of states merely changes smoothly as the period varies. An important fact should still be noted: the bosonization approach predicts, a constant ΔE in this case. We see here that this does not hold over the entire energy range: the period varies smoothly over large energy scales, and so does the average energy spacing; it increases with the energy.

Secondly, for $\lambda > 1$ there is an energy, in fact the separatrix, being situated for K at the value -1, where the period $T(E)$ *diverges*. To estimate the average energy spacing in this region, we consider trajectories near the classical separatrix where they spend a long time near the unstable equilibrium, that is, in a region in which ϕ

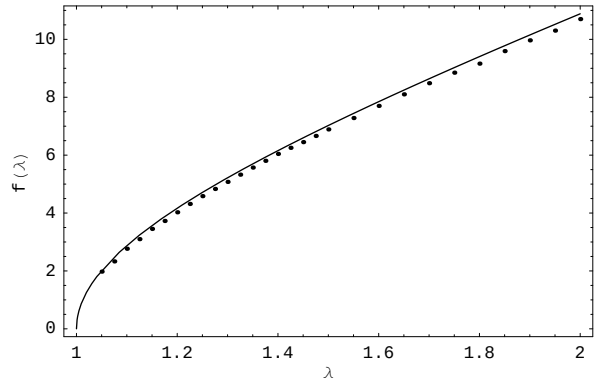

FIG. 1: Asymptotic behavior of the distance of levels of one parity at the transition point as a function of λ. The points are the numerical fits for $500 < N < 1500$ when fitted to $f(\lambda)/\ln(N)$; the solid curve is $2\pi\sqrt{\lambda^2-1}$.

as well as μ are small. One has for K approximately

$$K = -1 + \frac{1+\lambda}{2}\phi^2 - \frac{\lambda-1}{2}\mu^2. \tag{12}$$

Since this dominates the divergence of $T(E)$ near $E = -1$, one finds that $T(E)$ is approximately given by $\ln|2E/N+1|/\sqrt{\lambda^2-1}$. But $2E/N+1$ is itself of the order $2\Delta E/N$ and hence of first order in $1/N$. We thus obtain

$$\Delta E = \frac{2\pi\sqrt{\lambda^2-1}}{\ln N}. \tag{13}$$

This corresponds to the high density of states observed in [8] for specific values of $\lambda > 1$. It was found that a change occurred between the two regimes below and above a certain λ-dependent energy $E_c(\lambda)$: for low energies $E < E_c(\lambda)$, the states were deformed, the order parameter was non-zero and an odd-even degeneracy was observed. For $E > E_c(\lambda)$, all these phenomena disappeared and a normal regime, similar to $\lambda < 1$ was recovered. The transition region between those two regimes had the typical signature of high density of states. From (11) it becomes clear that the lower portion of the two regimes correspond to bounded motion in one well breaking parity symmetry. We mention that the tunneling between the left hand and right hand wells determines the splitting to be $\sim \exp(-\text{const}/\hbar) \sim \exp(-\text{const}\,N)$. For higher energies, the corresponding classical motion is above the wells and symmetry is restored. The two different regimes are separated by the separatrix with its high density of states. Formula (13) has been verified numerically as shown in Fig. 1. The region of high density is predicted to occur at energy -1 corresponding to energies above the ground state around $(\lambda+\lambda^{-1}-2)/2$ which again is verified numerically.

As the third major result, (11) provides the leading analytic behavior of the spectrum at the transition point ($\lambda = 1$). Since the Hamiltonian is quartic for (classically) low energies, one can evaluate ΔE using the formula for $T(E)$ for a quartic oscillator valid for energies high up in the spectrum but still small with respect to N. It is given by $T(E) = \text{const.} \cdot E^{-1/4}$, from which

$$\Delta E = \text{const.} \cdot (E/N)^{1/4} \tag{14}$$

follows, and it entails

$$E_k \sim k^{4/3}/N^{1/3}. \tag{15}$$

The $N^{-1/3}$ behavior has been obtained recently [7] and the $k^{4/3}$ behavior is confirmed numerically (see Fig.2). We stress the non-uniform nature of the limit played by the critical point $\lambda = 1$ when comparing (14) and (13).

It is obvious that the spectra of the two Hamiltonians (1) and (2) differ. This is due to the issue of ordering: in order to make sense of (2), we must specify in which way we order μ and ϕ to obtain a self-adjoint operator. There is no unique prescription for this so that an unknown difference exists between the two Hamiltonians. It is, however, known that if care is taken, these errors are of order $\hbar^2$, that is, of order N^{-2}. From this follows that we also expect the singularities to be different. We recall that they are associated with the critical point and the transitions for $\lambda > 1$. As the semi-classical treatment preserves these basic features it is expected that at least the qualitative pattern of the exceptional points remains. In fact, there is (i) the special feature at $\lambda = 1$ (an accumulation point for $N \to \infty$ [8]), (ii) a high density of EPs near to the separatrix [15], that is for energies around $E_c(\lambda)$, (iii) the absence of singularities near to real values for $\lambda < 1$, and for $\lambda > 1$ for energies sufficiently distant from $E_c(\lambda)$.

Of interest in the semi-classical treatment is the behavior of the wave function at the phase transition for $\lambda > 1$. In accordance with the long dwelling time classically at the saddle point, there is the phenomenon of super-scarring for the wave function. In fact, this has been shown generally [20] for the occurrence of such double wells. It arises for the specific values of k where the separatrix itself satisfies the WKB condition (10). The wave function then shows a dramatic concentration at the saddle-point of the Hamiltonian (or at the maximum in configuration space): in particular, it can be shown that there is an interval, the length of which goes to zero with $\hbar$, that is with $1/N$, in which the whole wave function is asymptotically concentrated. This is reflected in the eigenvectors of (1) associated with the eigenvalues at the minimal gap: in the basis of J_z they become relatively concentrated in the first few components, that is around $\mu = 0$; while the number of substantial components increases with N, the first twenty components exhaust the norm by about 50% irrespective of N. This has been tested for $\lambda = 1.1$, 1.5 and 2.0 where the transition occurs at $k \approx N/120$, $N/16$ and $N/8$, respectively.

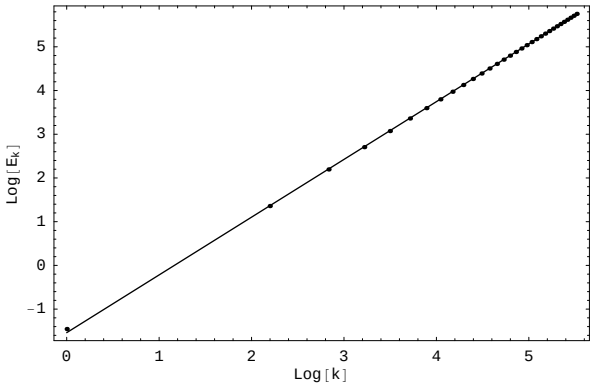

FIG. 2: Log-log-plot of E_k versus k. For clarity only every eighth point of the first 500 levels are taken ($N = 5000$). The straigth line fits the slope 4/3 with 1% accuracy.

The effect is rather significant – the more so the nearer λ to unity – in that the first twenty components of the neighboring wave functions contribute appreciably less to the total norm. In principle, wave functions can be directly determined semi-classically. As this is not the major focus of this paper we only outline the procedure. Switching to the usual polar coordinates defined around the z-axis but keeping the definition of μ as before, the renormalized Hamiltonian K reads

$$K = \mu + \frac{\lambda}{2}(1-\mu^2)\cos 2\phi. \tag{16}$$

In these variables, the usual spherical harmonics are eigenfunctions of the operator $\hat{\mu} = (2/iN)\partial/\partial\phi$ with an eigenvalue proportional to that of J_z. The semi-classical eigenfunctions of (16) in the eigenbasis of $\hat{\mu}$ are then expressed as a function of the energy by means of standard WKB formulae for the one-dimensional Hamiltonian (16). The expressions fail, of course, in the usual manner near the turning points of (16).

To summarize: using the semi-classical version of the original model (1) leads to (2).The WKB approximation then yields the expansion (8). Most information can then be extracted from (11) being based on (10). There is (i) the qualitative result about the phase transition occurring for $\lambda > 1$ as discussed in [8]. This includes the deviation from the strict equidistant level sequence for large N and for $\lambda < 1$ as well as the exponential (in N) separation of the degenerate levels for $\lambda > 1$. There is (ii) the expression (13) for the level distance at the transition point for $\lambda > 1$; there is (iii) the finding (15) at $\lambda = 1$. The apparent contrast of results (ii) and (iii) underlines once again the nonuniform nature of the large N limit at $\lambda = 1$. Note that all these results have been confirmed numerically. Additional results referring to the semi-classical wave function, in particular the superscarring at the saddle point in phase space are presented. The qualitative behavior of the singularities of the spectrum (the exceptional points) seems to be preserved in the semi-classical approach.

Acknowledgment The authors are grateful to Hannes Kriel for preparation of the drawing. One of us (WDH) gratefully acknowledges the generous hospitality that he enjoyed during his three weeks stay from Thomas Seligman, Director of the CIC, and the colleagues at the Department of Physics at Cuernavaca, Mexico.

[1] H.J. Lipkin, N. Meshkov and N. Glick, Nucl. Phys. A **62**, 188 (1965); *ibid.* **62**, 199 (1965), *ibid.* **62**, 211 (1965)
[2] R. Botet, R. Jullien and P. Pfeuty, Phys. Rev. Lett. **49**, 478 (1982)
[3] J. I. Cirac, M. Lewenstein, K. Mølmer, and P. Zoller, Phys. Rev. A **57**, 1208 (1998)
[4] J. Vidal, G. Palacios, and R. Mosseri, Phys. Rev. A **69**, 022107 (2004)
[5] F. Pan and J. P. Draayer, Phys. Lett. B **451**, 1 (1999)
[6] J. Links, H.-Q. Zhou, R. H. McKenzie, and M. D. Gould, J. Phys. A **36**, R63 (2003)
[7] S. Dusuel and J. Vidal, Phys. Rev. Lett. **93**, 237204 (2004), and papers quoted therein
[8] W.D. Heiss, F.G. Scholtz and H.B. Geyer, J. Phys. A **38**, 1843 (2005)
[9] F.G. Scholtz, B.H. Bartlett and H.B. Geyer, Phys. Rev. Lett. **91**, 80602 (2004); J.N. Kriel, A. Morozov and F.G. Scholtz, J. Phys. A **38**, 205 (2005)
[10] W.D. Heiss and A.L. Sannino, Phys. Rev. A **43**, 4159 (1991)
[11] T. Kato, *Perturbation theory of linear operators* (Springer, Berlin, 1966)
[12] C.N. Yang and T.D. Lee, Phys. Rev. **87**, 410 (1952)
[13] P. Borrmann, O. Mülken and J. Harting, Phys. Rev. Lett. **84**, 3511 (2000)
[14] P. Cejnar, S. Heinze and J. Dobes, *preprint* nucl-th/04-06060
[15] W.D. Heiss and M. Müller, Phys. Rev. E **66**, 016217 (2002)
[16] P. Ring and P. Schuck, *The Nuclear Many Body Problem*, Springer, New York (1980)
[17] The unusual convention used by this choice avoids the coincidence of the physically important South pole (see below) with the singular point of the polar coordinates.
[18] J. Schliemann and F.G. Mertens, J. Phys. C **10**, 1091 (1998)
[19] M.Z.I. Gering and W.D. Heiss, Phys. Rev. C **29**, 1113 (1984); W.D. Heiss and R.H. Lemmer, Phys. Rev. C **29**, 2368 (1984)
[20] Y. Colin de Verdière and B. Parisse, Comm. Partial Differential Equations **19**, 1535 (1994); Ann. Institut Henri Poincaré **61**, 347 (1994)